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The Algebra of Propositions

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IT IS proposed in this paper to develop a method by which the most general problem of the algebra of propositions is solved. This problem is to construct all propositions whose truth is independent of the form of the variables. As might be expected this method will enable us to determine without the use of matrices the consistency and independence of propositions, except in the case of those fundamental properties, which taken together define consistency itself. In the discussion which follows no limitations are placed on the meaning of the terms and the word proposition is used simply in the sense of a variable. If we were to say, for example, that our variables must be restricted to zero or one values, we would be introducing a limitation which would reduce the generality of our science. A logic may be false in the sense that its postulates lead to contradictory theorems, or it may be false in the sense that it lacks generality, and the situation is not saved by calling it a two-valued, or a three-valued logic, or a logic of "propositions," or by designating it by some other euphuism.

Because of the length of the paper we shall not be able to include at this time a critical discussion of the foundations on which the system rests, but the assumptions that we make, whether they turn out on examination to be fundamental or derived, will be set

down as we make them. The first pages will largely repeat what has been said in Part I of a previous article, *Abstract Logic or the Science of Modality* (Phil. of Sci. Vol. I, No. 4, Oct. 1934), because what is there is essential to an understanding of what follows here. In that article the analogous problem is stated and solved for the Aristotelian algebra of classes. The general solution of the Boolean algebra of classes requires no special ingenuity or novelty of method.

It is important to notice that a propositional form is not shown to be unessential to our algebra because it can be expressed in terms of some other form. Thus, conjunction and disjunction can each be expressed in terms of the other but are differentiated by the properties which our expansion formulas express. In our algebra the following propositional forms are recognized as necessary and sufficient for the expression of any truth:

The form of negation:
p' = p is untrue.

The hypothetical form:
$p \angle q$ = p implies q,
$(p \angle q)'$ = p does not imply q.

The conjunctive form:
$p \cdot q$ = p (is true) and q (is true).

The disjunctive form:
$p + q$ = either p (is true) or q (is true).

The null-proposition will be defined as a p that satisfies the implication,

$$p \angle p'$$

and will be represented by the symbol o.

Its negative, the one-proposition, will be defined as a p that satisfies

$$p' \angle p$$

and will be represented by the symbol i.

Propositions, i. e. propositional functions, will be represented by the letters $p, q, r, \ldots$ Let us *provisionally* interpret the expression $|p, q, r, \ldots|$ as the assertion, true or false:

"There exist specific meanings of the terms that enter into $p, q, r, \ldots$ that make $p, q, r, \ldots$ simultaneously true."

The expression $|p, q, r, \ldots|$ is called a modal function of the n elements $p, q, r, \ldots$ It is said to be of the nth. degree and, if none of its elements are governed by a modal operator, to be of the first order. The simple properties of modal functions, as well as the formulas for their expansion, were first stated by the author in the *Journal of Philosophy* 1926, the formulas for the expansion of functions of higher order in his *Symbolic Logic* 1927. The relation of any variable p to any modal function of p follows at once from the assumptions:

Definition:

$$|p| = (p \angle o)'$$

Postulate:

$$p \angle (p \angle o)'$$

Principle:

$$p \angle ||p|'|'_n \cdot \angle \cdot p \angle ||p|'|'_{n+1}$$

When we wish to indicate that the modal operator is repeated we write:

$$|p|_2 = ||p||, |p| = ||p|'|'_0 = |p|'', |p|_3 = |||p||| \text{ etc.}$$

Thus, making $n = o$ in the principle there results,

$$p \angle |p| \cdot \angle \cdot p \angle ||p|'|'$$

so that we have generally,

Theorem:

$$p \angle ||p|'|'_n$$

Substituting in succession for n the values o, 1, 2, 3, it follows that

$$p \angle p|, p \angle p|'|', p \angle p|'||', p \angle p|'|||'$$

leaving out the first half of the operator to save space. If we call these 1, 2, 3, 4, respectively, then by

(a) $(p \angle q) \angle (q' \angle p')$

$p \angle p\|'\|\|'$	
$p\|'\|\|' \angle p\|'\|\|'$	$p = p\|'\|\|$ in (1) then (a)
$p\|'\|\|' \angle p\|'\|\|'\|$	$p = p\|'\|\|'$ in (1)
$p\|'\|\|'\| \angle p\|'\|'$	$p = p\|$ in (2) then (a) and $p = p\|'$
$p\|'\|' \angle p\|'\|'\|'\|'$	$p = p\|'$ in (2) then (a) and $p = p\|'$ then (a)
$p\|'\|'\|'\|' \angle p\|'\|'\|$	$p = p\|'$ in (1) then (a) and $p = p\|'\|'$
$p\|'\|'\| \angle p\|'\|'\|\|$	$p = p\|'\|'\|$ in (1)
$p\|'\|'\|\| \angle p\|$	$p = p\|'$ in (3) then (a)
$p\| \angle p\|\|'\|\|'$	$p = p\|$ in (3)
$p\|\|'\|\|' \angle p\|\|'\|'$	$p = p\|$ in (1) then (a) and $p = p\|\|'$
$p\|\|'\|' \angle p\|\|'\|'\|$	$p = p\|\|'\|'$ in (1)
$p\|\|'\|'\| \angle p\|\|$	$p = p\|\|'$ in (2) then (a)
$p\|\| \angle p\|\|\|'\|'$	$p = p\|\|$ in (2)
$p\|\|\|'\|' \angle p\|\|\|$	$p = p\|'$ in (1) then (a) and $p = p\|\|$
$p\|\|\| \angle p\|\|\|\|$	$p = p\|\|\|$ in (1)

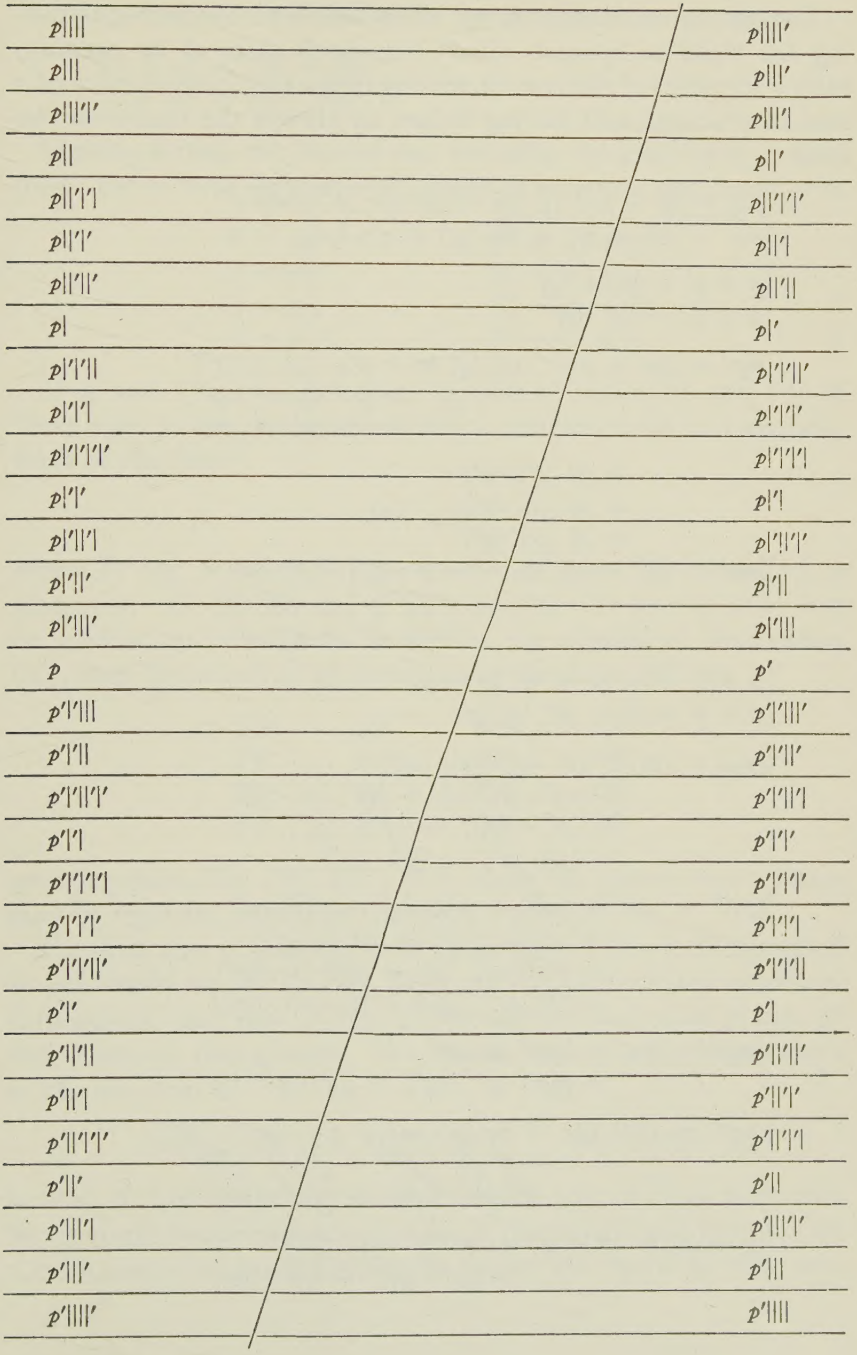
The other half of the table which we give below is obtained by changing p into p' throughout. In the table we symbolize p and the different modals of p by "lengths", the line down the center terminating these lengths and separating each modal from its contradictory. Thus $p \angle p|$ would read "the length p is contained in the length $p|$." It may now be established as in the paper referred to above (*Abstract Logic* etc. pp. 373–375) that the converse of each one of the implications set forth above is untrue. The table evidently extends to infinity in both directions as it must in any logic which recognizes all modal distinctions.

In order to attack the general problem whose solution is given in the sequel, it will be necessary to recognize a new but fundamental principle, which expresses the modal of the product of two variables as a sum of products of modal functions of the single variables. This formula is:

$$|p \cdot q| = pq + p'q' \cdot |p| \cdot |q| + |p| \cdot |q'|' + |q| \cdot |p'|'$$

It fails (does not necessarily hold) when a single one of the quadrants pq, $p'q$ or pq' vanishes and holds in all other cases. The cases in which it fails happen to be the very cases in which it is not needed for,

$$\begin{aligned} pq \angle 0 \cdot \angle \cdot |pq| &= 0 \\ pq' \angle 0 \cdot \angle \cdot |pq| &= |p| \\ p'q \angle 0 \cdot \angle \cdot |pq| &= |q| \end{aligned}$$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The other formulas set down below follow as theorems, assuming the right to operate with | | on both sides of an *equality*, taking for granted the law for writing down the contradictory of a sum of products and having before us always the results of the table.

$$|pq| = pq + p'q' \cdot |p| \cdot |q| + |p| \cdot |q'|' + |q| \cdot |p'|'$$
$$|pq|' = |p|' + |q|' + p'q \cdot |q'| + q'p \cdot |p'|$$

$$|p + q| = |p| + |q|$$
$$|p + q|' = |p|' \cdot |q|'$$

$$|pqr| = pqr + p'q'r' \cdot |p| \cdot |q| \cdot |r| + p'q' \cdot |p| \cdot |q| \cdot |r'|' + p'r' \cdot |p| \cdot |r| \cdot |q'|' + q'r' \cdot |q| \cdot |r| \cdot |p'|' + |p| \cdot |q'|' \cdot |r'|' + |q| \cdot |p'|' \cdot |r'|' + |r| \cdot |p'|' \cdot |q'|'$$
$$|pqr|' = |p|' + |q|' + |r|' + p'q \cdot |q'| + p'r \cdot |r'| + q'p \cdot |p'| + q'r \cdot |r'| + r'p \cdot |p'| + r'q \cdot |q'|$$

$$|p + q + r| = |p| + |q| + |r|$$
$$|p + q + r|' = |p|' \cdot |q|' \cdot |r|'$$

$$||pq|| = |pq| + |p|' \cdot |q|' \cdot ||p|| \cdot ||q|| + p|p'| \cdot |q|' \cdot ||q|| + q|q'| \cdot |p|' \cdot ||p|| + ||p'|'| \cdot ||q|'|' + ||q'|'| \cdot ||p|'|' + ||p|| \cdot ||q'|| + ||q|| \cdot ||p'||$$

$$||pq||' = ||p||' + ||q||' + p'|p| \cdot |q|' + q'|q| \cdot |p|' + pq'|q| \cdot ||p'|'|' + qp'|p| \cdot ||q'|'|' + p|p'| \cdot |q| \cdot ||q'|| + q|q'| \cdot |p| \cdot ||p'|| + |p|' \cdot |q'|' \cdot ||q'|| + |q|' \cdot |p'|' \cdot ||p'||$$

$$||pq|'| = |pq|' + ||p|'| + ||q|'| + p|p'| \cdot |q'|' \cdot ||q'|| + q|q'| \cdot |p'|' \cdot ||p'||$$

$$||pq|'|' = |p'|' \cdot |q'|' + pq \cdot |p'| \cdot |q'| + p'q' \cdot ||p|'|' \cdot ||q|'|' + p'|q'|' \cdot ||p|'|' + q'|p'|' \cdot ||q|'|' + ||p|'|' \cdot ||q'||' + ||q|'|' \cdot ||p'||'$$

$$||p + q|'| = |p + q|' + |p| \cdot |q| \cdot ||p|'| \cdot ||q|'| + ||p|'| \cdot ||q||' + ||q|'| \cdot ||p||'$$

$$||p + q||' = ||p||' + ||q||' + |p| \cdot |q|' \cdot ||q|| + |q| \cdot |p|' \cdot ||p||$$

$$||p + q|| = ||p|| + ||q||$$
$$||p + q||' = ||p||' \cdot ||q||'$$

Having solved the case of one variable, we shall have to deal from now on with eight atomic modes of assertion and denial, viz.

$$\begin{array}{ll} P = pq & P' = p' + q' \\ \mathcal{Q} = pq' & \mathcal{Q}' = p' + q \\ R = p'q & R' = p + q' \\ S = p'q' & S' = p + q \end{array}$$

If X and Υ are conceived as capable of taking on any one of these eight values, we wish initially to determine all true propositions of the form:

$$MX \angle N\Upsilon$$

where M and N are variables which may have any modal value whatever. If the M and N are omitted for convenience but understood to be present, as well as the symbol of implication ($\angle$), then the set of all propositions of this form will be:

$$\begin{array}{llll} PP & \mathcal{Q}P & RP & SP \\ P\mathcal{Q} & \mathcal{Q}\mathcal{Q} & R\mathcal{Q} & S\mathcal{Q} \\ PR & \mathcal{Q}R & RR & SR \\ PS & \mathcal{Q}S & RS & SS \end{array}$$

omitting pairs like $P\mathcal{Q}'$, $\mathcal{Q}R'$, $R'S'$, since the prime over a letter may be regarded as part of the modal operator.

Consider now some member of the set, say $P\mathcal{Q}$, which expressed in full would be $MP \angle N\mathcal{Q}$, $Mpq \angle Npq'$, and change p into p' throughout, and then q into q' throughout, and then p into p' and q into q' throughout. We should then obtain three others of the set, viz.

$$MR \angle NS \qquad M\mathcal{Q} \angle NP \qquad MS \angle NR$$

so that if there were any cases in which any of these were true there would be corresponding cases in the others, and it would be a mere matter of book keeping to record the variants when one

had been found. We may then omit these three cases from the set and confine our attention to the first.

On making these substitutions systematically it would turn out that we should only have to consider the pairs in the first column, PP, $P\mathcal{Q}$, PR, PS. Written out in full these would be:

$$Mpq \angle Npq \qquad Mpq \angle Npq' \qquad Mpq \angle Np'q \qquad Mpq \angle Np'q'$$

If M and N are affirmative modal operators the last three types are seen to be invalid on making $p = q = i$; if M and N are negative on making $p = q = o$ in PS, $p = o$, $q = i$ in PR, $p = i$, $q = o$ in $P\mathcal{Q}$. Similarly, if N is affirmative and M negative, all propositions of the form $MP \angle N\mathcal{Q}$ are seen to be invalid on making $p = o$, on making $q = o$ all of the form $MP \angle NR$, on making $p = q'$ all of the form $MP \angle NS$. Omiting the case $Mpq \angle Npq$ which is already solved, it remains to consider:

M affirmative, N negative, the cases

$$Mpq \angle Npq' \qquad Mpq \angle Np'q \qquad Mpq \angle Np'q'$$

In order to illustrate the method of obtaining the true members of any set, consider the mood $P'||' \angle R'|'|'$ or $P'||' \cdot R'|'| \angle o$.

$$P'||' = ||p' + q'||' = ||p'||' \cdot ||q'||'$$
$$R'|'| = ||p|' \cdot |q|'|$$

1. If $|p|' \cdot |q|' = o$ $\qquad P'||' \cdot R'|'| = o$

2. If $|p| \cdot |q|' = o$ since $|q|' \angle |p|'$ $\qquad P'||' \cdot R'|'| = ||p'||' \cdot ||q'||' \cdot ||q|'| = o$

3. If $|p|' \cdot |q| = o$ since $|p|' \angle |q|'$ $\qquad P'||' \cdot R'|'| = ||p'||' \cdot ||q'||' \cdot ||p|'| = o$

4. If $|p|' \cdot |q|' = o$ or if none of the quadrants of the universe vanishes so that the expansion formulas hold,

$$R'|'| = |p|' \cdot |q|' + |p| \cdot |q| \cdot ||p|'| \cdot ||q|'| + ||p|'| \cdot ||q||' + ||q|'| \cdot ||p||'$$
and $||P'||' \cdot ||R'|'| = o$

Suppose again the mood: $P'||' \angle R|'|$ or $P'||' \cdot R|'|' \angle 0$.

$$P'||' = ||p'||' \cdot ||q'||' \qquad R|'|' = ||p'q'|'|'$$

1. If $p'q' = 0$ $\qquad P'||' \cdot R|'|' = 0$

2. If $pq' = 0$ $\qquad P'||' \cdot R|'|' = 0$

3. If $p'q = 0$ $\qquad P'||' \cdot R|'|' = 0$

4. If $pq = 0$ or if none of the quadrants vanishes,

$$\begin{aligned} ||p'q'|'|' = {} & |p|' \cdot |q|' + p'q' \cdot |p| \cdot |q| + pq \cdot ||p'|'|' \cdot ||q'|'|' \\ & + p|q|' \cdot ||p'|'|' + q|p|' \cdot ||q'|'|' \\ & + ||p'|'|' \cdot ||q|| + ||q'|'|' \cdot ||p|| \end{aligned}$$

and $P'||' \cdot R|'|' = 0$ as before.

We have thus established the truth of the propositions in question, $||p' + q'||' \angle ||p + q|'|'$ and $||p' + q'||' \angle ||p'q'|'|$.

In general we should have:

$$\begin{aligned} ||P'||' &= ||p' + q'||' = ||p'||' \cdot ||q'||' \\ |P'|' &= |p' + q'|' = |p'|' \cdot |q'|' \\ ||P'|'| &= ||p' + q'|'| = ||p'|' \cdot |q'|'| \end{aligned}$$

1. If $|p'|' \cdot |q'|' = 0$ then $||P'|'| = 0$

2. If $|p'|' \cdot |q'| = 0$ then $||P'|'| = ||p'|'|$

3. If $|p'| \cdot |q'|' = 0$ then $||P'|'| = ||q'|'|$

4. If $|p'| \cdot |q'| = 0$ or if none of the four quadrants vanishes

$$\begin{aligned} ||P'|'| = {} & |p'|' \cdot |q'|' + |p'| \cdot |q'| \cdot ||p'|'| \cdot ||q'|'| \\ & + ||p'||' \cdot ||q'|'| + ||q'||' \cdot ||p'|'| \end{aligned}$$

$P = pq$

$$||P|'|' = ||pq|'|'$$

1. If $pq = 0$ then $||P|'|' = 0$

2. If $pq' = 0$ then $||P|'|' = ||p|'|'$

3. If $p'q = 0$ then $||P|'|' = ||q|'|'$

4. If $p'q' = o$ then $||P|'|' = |p'|' \cdot |q'|' + pq \cdot |p'| \cdot |q'| + p'q' \cdot ||p|'|' \cdot ||q|'|'$

$$+ p' \cdot |q'|' \cdot ||p|'|' + q' \cdot |p'|' \cdot ||q|'|' + + ||p|'|' \cdot ||q'||' + ||q|'|' \cdot ||p'||'$$

which also holds when none of the four quadrants vanishes as well as in all limiting cases.

$|P| = |pq|$

1. If $pq = o$ then $|P| = o$
2. If $pq' = o$ then $|P| = |p|$
3. If $p'q = o$ then $|P| = |q|$
4. If $p'q' = o$ or if none of the four quadrants vanishes

$$|P| = pq + p'q' \cdot |p| \cdot |q| + |p| \cdot |q'|' + |q| \cdot |p'|'$$

$||P|| = ||pq||$

1. If $pq = o$ then $||P|| = o$
2. If $pq' = o$ then $||P|| = ||p||$
3. If $p'q = o$ then $||P|| = ||q||$
4. If $p'q' = o$ or if none of the four quadrants vanishes as well as in all limiting cases,

$$||P|| = |pq| + |p|' \cdot |q|' \cdot ||p|| \cdot ||q|| + p \cdot |p'| \cdot |q|' + q \cdot |q'| \cdot |p|' + ||p'|'| \cdot ||q|'|' + ||q'|'| \cdot ||p|'|' + ||p|| \cdot ||q'||' + ||q|| \cdot ||p'||'$$

To obtain the corresponding values of $\mathfrak{Q}$, R, S, we change q into q', p into p', and then q into q' and p into p' throughout. The products that vanish may now be found by direct multiplication.

It is important to notice that we are not permitted to operate on both sides of an implication by the same modal operator. Thus,

$$p \prec ||p'|' \text{ but}$$
$$p| \prec |||p|'|'| \text{ is false}$$

If, however, $Lp \angle Lq$ and $Np \angle Nq$ and if M is an operator occurring in the modal scale (that is, in the table) between L and N, then $Mp \angle Mq$. Thus, if $q = p|'|'|$ and $Lp = p'|'$, $Np = p$, $Mp = p'|'|'|$, we have

$$\begin{aligned} |p'|' &\angle ||||p|'|'|'|' \\ ||p'|'| &\angle |||||p|'|'|'|'| \\ p &\angle |||p|'|'| \end{aligned}$$

This principle is true only when the conditions are given categorically, not as a hypothetical expressed symbolically,

$$(Lp \angle Mp)\cdot(Mq \angle Nq)\cdot(Lp \angle Lq)\cdot(Np \angle Nq) \angle (Mp \angle Mq)$$

As an illustration of the method consider the product $||P'|'|\cdot||S'|'|$.

$||P'|'| = ||p'|'\cdot|q'|'|$

1. If $|p'|'\cdot|q'|' = 0$, $||P'|'| = 0$
2. If $|p'|'\cdot|q'| = 0$, $||P'|'| = ||p'|'|$
3. If $|p'|\cdot|q'|' = 0$, $||P'|'| = ||q'|'|$
4. If $|p'|\cdot|q'| = 0$, or if none of the quadrants vanishes, the expansions hold and

$$\begin{aligned} ||P'|'| = {} & |p'|'\cdot|q'|' + |p'|\cdot|q'|\cdot||p'|'|\cdot||q'|'| \\ & + ||p'||'\cdot||q'|'| + ||q'||'\cdot||p'|'| \end{aligned}$$

$||S'|'| = ||p|'\cdot|q|'|$

1. If $|p|'\cdot|q|' = 0$, $||S'|'| = 0$
2. If $|p|'\cdot|q| = 0$, $||S'|'| = ||p|'|$
3. If $|p|\cdot|q|' = 0$, $||S'|'| = ||q|'|$
4. If $|p|\cdot|q| = 0$, or if none of the quadrants vanishes

$$\begin{aligned} ||S'|'| = {} & |p|'\cdot|q|' + |p|\cdot|q|\cdot||p|'|\cdot||q|'| \\ & + ||p||'\cdot||q|'| + ||q||'\cdot||p|'| \end{aligned}$$

Let us consider condition 1 for the value of $||P'|'|$ in conjunction with each one of the conditions 1, 2, 3, 4, for the value of $||S'|'|$.

These cases we may designate by the notation 1.1, 1.2, 1.3, 1.4. But since these as well as 2.1, 3.1, 4.1, lead to a vanishing of the product $||P'|'|\cdot||S'|'|$, they may be omitted. Consider, then, the other combinations:

2.2 $||P'|'|\cdot||S'|'| = ||p'|'|\cdot||p|'| = 0$

2.3 $||P'|'|\cdot||S'||' = ||p'|'|\cdot||q|'| = 0$

for from the conditions given we have

$|q'| \angle |p'|$ and $|q|' \angle |p|'$, so that
$||q|'| \angle ||p|'|$, $||q'|| \angle ||p'|'|'$, $||q|'|\cdot||p'|'| \angle 0$

2.4 $||P'|'|\cdot||S'|'| = 0$ since
$|p|'\cdot||p'|'| = 0$, $||p|'|\cdot||p'|'| = 0$

3.2 $||P'|'|\cdot||S'|'| = 0$ since
$||q'|'|\cdot||p|'| = 0$ as in 2.3

3.3 $||P'|'|\cdot||S'|'| = 0$ since
$||q'|'|\cdot||q|'| = 0$

3.4 $||P'|'|\cdot||S'|'| = 0$ since
$|q|'\cdot||q'|'| = 0$, $||q|'|\cdot||q'|'| = 0$

4.2 $||P'|'|\cdot||S'|'| = 0$ since
$|p'|'\cdot||p|'| = 0$, $||p'|'|\cdot||p|'| = 0$, $||p'||'\cdot||p|'| = 0$.

4.3 $||P'|'|\cdot||S'|'| = 0$ similarly.

4.4 $||P'|'|\cdot||S'|'| = 0$ similarly.

The most general form of a proposition in the variables p, q, r, ... would be an expression in which the signs of operation $+$ and $\times$, as well as the variables and the modal operators, were ordered and punctuated in such a way as to be interpretable. Once interpretable such an expression or any of its parts can be viewed either as a sum of products or as a product of sums. It is clear that a product of sums can be reduced to a sum of products and that for either of these to be true its contradictory must vanish. Moreover, its contradictory can always be put in the form of a sum of products. Therefore the most general problem

of the algebra of propositions can be reduced to that of determining all the cases in which a sum of products of modal functions of a set of variables whose form is undetermined, vanishes. Since for a sum to vanish each part must vanish separately, the most general problem of the algebra of propositions may now be stated in its simplest form: To determine all the cases in which a product of modal functions, into which an indefinite number of variables enters, vanishes. The truth or untruth of any specific proposition viewed as of this or as of any other form may be determined by the method we develop, it being understood that we are dealing only with logical symbols and that the form of the variables is entirely undetermined.

From the very general point of view which is here developed, some of the formulas recognized in the classical or Boolean algebra turn out to be true only under restriction. Thus we can not write,

$$|pq'|' \cdot |qr'|' \angle |pr'|'$$

as appears from the following consideration:

$$p \angle q \cdot q \angle r \cdot \angle \cdot p \angle r$$

gives for $q = ||p|'|', r = 0$

$$p \angle ||p|'|' \cdot ||p|'|' \angle 0 \cdot \angle \cdot p \angle 0$$

from which, suppressing the first premise,

$$|p| \cdot |||p|'|'|' \angle 0, \text{ which is untrue.}$$

Again,

$$p \angle q \cdot r \angle s \cdot \angle \cdot pr \angle qs$$

expressed in modal notation is,

$$|pq'|' \cdot |rs'|' \cdot [|pq'r| + |prs'|] \angle 0$$

and since for a sum to vanish the parts must vanish separately, we must have, changing pq' into p, a procedure involving no loss of generality,

$$|p|' \cdot |rs|' \cdot |pr| \angle 0$$

Now substitute $s = p'$, $r = p$, $p = ||p|'|'$, and we have,

$$|p| \cdot |||p|'|'|'|' \angle 0, \text{ which is untrue.}$$

And so again for some of the theorems dependent on these principles:

$$pq \angle r \cdot s \angle p \cdot \angle \cdot sq \angle r \text{ or } pq \angle 0 \cdot s \angle p \cdot \angle \cdot sq \angle 0$$

fails for

$$p = s|'|', q = 1$$

($|pq|' \cdot |sp'|'|sq|$ then becomes $|||s|'|'|' \cdot 1 \cdot |s|$, which does not vanish).

If each affirmative modal is either contained in or contains every other affirmative modal, then a simple proof can be given that the law of transitivity fails in any logic in which all modal distinctions are recognized. Thus,

Assume 1. $p \angle ||p|'|'$

Now $p \angle ||p|'|' \cdot ||p|'|' \angle 0 \cdot \angle \cdot p \angle 0$

from which, suppressing the first premise,

$|||p|'|'|' \angle |p|'$

Now substitute $p = |p|'$ in 1, then,

$|p|' \angle |||p|'|'|'$

so that

$|p|' = |||p|'|'|'$

Assume 2. $||p|'|' \angle p$

$||p|'|' \angle p \cdot p \angle 0 \cdot \angle \cdot ||p|'|' \angle 0$

from which, suppressing the first premise,

$|p|' \angle ||p|'|'|'$

Now substitute $p = |p|'$ in 2, then,

$|||p|'|'|' \angle |p|'$

so that

$|p|' = |||p|'|'|'$

The so-called two-valued logic, which recognizes no modal distinctions at all, may be said to possess a minimum generality. It illustrates the fact, seemingly not obvious to our contemporaries, that a logic can be consistent and yet false. It is not surprising, therefore, that it should find itself helpless in the face of any paradox, whose solution depends on making the proper modal distinctions. Nor is it surprising that certain simple laws, like the law of transitivity, should fail in a logic that is perfectly general, but become true in the very restricted case developed in *Principia Mathematica.*

A logic like the Boolean which assumes transitivity, the right to strengthen a promise or weaken a conclusion etc., may be said to hold on a special matrix, a matrix in which certain modal distinctions are wiped out. Thus if our table represents the general matrix which recognizes all modal distinctions, the special or Boolean matrix would wipe some of these out. The matrix of the "two-valued" logic would be one in which the dividing line of our table has become perpendicular to the base.

An important special case of the general problem we are considering and which in the sequel will turn out to be decisive, would be the one in which the terms are arranged in cyclical order,

$$lm\ mn \ldots pq\ qr \ldots xy\ yz \ldots 1$$

and wherein the modal operators occur in arbitrary positions,

$$1\ L\ m\ m\ n \ldots p\ q\ M\ q\ r \ldots x\ y\ y\ N\ z \ldots 1$$

More specifically, if the modal operators should occur at regular intervals of two variables apart, we should have,

$$Llm.\ Mmn.\ Nnp \ldots Pqr.\ Qrs \ldots Rzl$$

We have already determined all true propositions of the form,

$$Mpq.\ Nqp \prec o,$$

a cycle of two variables with the modal operators placed two variables apart, and wherein a prime may appear over the p or q in any position. Let us now determine all true propositions of the form,

$$Lpq.\ Mqr.\ Nrp \prec o$$

wherein a prime may appear over a p or a q or an r in any position. The set of all propositions of this form would be gotten by permuting the four letters, P, $\mathcal{Q}$, R, S, three at a time. Thus,

$LP.\ MP.\ NP \quad LP.\ MP.\ N\mathcal{Q} \quad LP.\ M\mathcal{Q}.\ NR \quad L\mathcal{Q}.\ MR.\ NS$ etc.

sixty-four cases in all. These sixty-four reduce to a very few if we employ the method that we used before. Thus in $pq\cdot\ qr\cdot\ rp$ substitute in succession $p = p'$, $q = q'$, $r = r'$, and we get RPR, $\mathcal{Q}RP$, $P\mathcal{Q}\mathcal{Q}$. Now substitute first $p = p'$ and $q = q'$, then $p = p'$ and $r = r'$, then $q = q'$ and $r = r'$, in succession, and we obtain SRR, $\mathcal{Q}S\mathcal{Q}$, $R\mathcal{Q}S$. Now prime all three letters and we get SSS. It is clear that once we have determined all true propositions of the form,

$$Lpq.\ Mqr.\ Nrp \angle o$$

it will be a mere matter of book keeping to record the cases that follow from these. If we begin with this array, it only remains to determine the nature of L, M, and N.

The array of all propositions of the form $Lpq.\ Mqr.\ Npr \angle o$.

Affirmative	Negative	Invalid by the substitution
L, M, N		$p, q, r, = i$
	L, M, N	$p, q, r, = o$
L	$M\ N$	$p, q, = i \quad r = o$
M	$L\ N$	$q, r, = i \quad p = o$
N	$L\ M$	$p, r, = i \quad q = o$
$L\ M$	N	See the discussion below
$L\ N$	M	Same as above
$M\ N$	L	" " "

If L and M are affirmative and N negative, we may prove the five cases from which all the others follow by "strengthening"

the factors. This strengthening of the factors is not permitted as a formal process but indicates that if the "critical" cases are true, the others are easily seen to follow in the same way.

1. $Z = ||(pq)'||' \cdot ||(qr)'||' \cdot ||(pr)'||$

$Z = ||p'||' \cdot ||q'||' \cdot ||r'||' \cdot [||p'|| + ||r'||] = 0$

2. $Z = |(pq)'|' \cdot |(qr)'|' \cdot |(pq)'|$

$Z = |p'|' \cdot |q'|' \cdot |r'|' \cdot [|p'| + |r'|] = 0$

3. $Z = pq.\ qr.\ (pr)'$

$Z = pq.\ qr.\ (pr)' = pqr.\ [p' + r'] = 0$

4. $Z = |(pq)'|' \cdot ||(qr)'||' \cdot ||(pr)'|'|'$

$Z = |p'|' \cdot ||q'||' \cdot ||r'||' \cdot ||(pr)'|'|'$

If 1. $pr = 0$, $|p'|' \cdot ||r'||' = 0$ and $Z = 0$

If 2. $pr' = 0$, $p \angle r$, and $Z = 0$ since $||(pr)'|'|' = ||p'|'|'$

If 3. $p'r = 0$, $r \angle p$, and $Z = 0$.

If 4. $p'r' = 0$, or if no quadrant of the universe vanishes,

$||(pr)'|'|' = ||p'|'|' + ||r'|'|' + |p'| \cdot |r'|' \cdot ||r'|| + |r'| \cdot |p'|' \cdot ||p'||$
since in this case the expansion holds and $Z = 0$.

5. $Z = ||(pq)'||' \cdot |(qr)'|' \cdot ||(pr)'|'|'$

$Z = ||p'||' \cdot ||q'||' \cdot |r'|' \cdot ||(pr)'|'|'$

If 1. $pr = 0$, $||p'||' \cdot |r'|' = 0$, and $Z = 0$.

If 2. $pr' = 0$, $p \angle r$, and $Z = 0$.

If 3. $p'r = 0$, $r \angle p$, and $Z = 0$.

If 4. $p'r' = 0$, or if no quadrant vanishes,
$Z = 0$ as before.

The invalidity of those cases that remain follows from the invalidity of the nine set down below by "weakening" the factors.

1. $Z = |(pq)'|' \cdot |(qr)'|' \cdot ||(pr)'||$

$Z = |p'|' \cdot |q'|' \cdot |r'|' \cdot [||p'|| + ||r'||] \neq 0.$

Note that we have tacitly assumed from the start that if p and q are independent variables, then $Lp \cdot Mq$ does not vanish unless one of the factors vanishes separately.

2. $Z = ||(pq)'|'| \cdot ||(qr)'||' \cdot |(pr)'|$

$||p'|'| \cdot ||q'||' \cdot ||r'||' \cdot [|p'| + |r'|] \neq 0$

3. $Z = ||(pq)'||' \cdot ||(qr)'|'| \cdot |(pr)'|$

$Z = ||p'||' \cdot ||q'||' \cdot ||r'|'| \cdot [|p'| + |r'|] \neq 0$

4. $Z = pq \cdot ||(qr)'||' \cdot ||(pr)'|'|'$

$Z = p \cdot ||q'||' \cdot ||r'||' \cdot ||(pr)'|'|'$

If $pr' = 0$, $p \angle r$, $Z = p \cdot ||q'||' \cdot ||r'||' \cdot ||p'|'|' \neq 0$

5. $Z = ||(pq)'||' \cdot qr \cdot ||(pr)'|'|'$

$Z = r \cdot ||p'||' \cdot ||q'||' \cdot ||(pr)'|'|'$

If $p'r = 0$, $r \angle p$, $Z = r \cdot ||p'||' \cdot ||q'||' \cdot ||r'|'|' \neq 0$

6. $Z = ||(pq)'|'| \cdot |(qr)'|' \cdot ||(pr)'|'|'$

$Z = |r'|' \cdot ||r'|'|' \neq 0$ for $p = q = i$

7. $Z = |(pq)'|' \cdot ||(qr)'|'| \cdot ||(pr)'|'|'$

$Z = |p'|' \cdot ||p'|'|' \neq 0$ for $q = r = i$

8. $Z = ||(pq)|'|' \cdot ||(qr)'||' \cdot ||(pr)||'$

$Z = ||p|'|' \cdot ||r'||' \cdot ||(pr)||'$ for $q = i$

For the case in which $||(pr)||'$ can be expanded $Z \neq 0$.

9. $Z = ||(pq)'||' \cdot ||(qr)|'|' \cdot ||(pr)||'$

$Z = ||p'||' \cdot ||r|'|' \cdot ||(pr)||'$ for $q = i$

For the case in which $||(pr)||'$ can be expanded $Z \neq o$.

The array $Lpq.\ Mqr.\ Npr' \angle o$

Affirmative	Negative	Invalid by the substitution
L, M, N		
	L, M, N	$p = q = r'$
L, M	N	$p = q = r$
L, N	M	$q = r'$
M, N	L	
L	M, N	
M	L, N	$p = o, q = r = i$
N	L, M	$q = o, p = r' = i$

Case I. L negative, M, N affirmative.

All the true propositions of this type have stronger premises than

$$||(pq)'||\cdot qr.\ pr' = o$$

and are valid as well.

The invalidity of all the others follows as in the two examples below:

1. $Z = ||pq||'\cdot||qr|'|'\cdot||(pr')'||'$. Let $p = q|'||, r = q|'||'$.

Note: $q||'|| \angle q|'$, $\therefore q|' \angle q||'||$ is false, $\therefore q|'|'\cdot q|'||'||' \angle o$ is false.

2. $Z = ||pq||'\cdot||(qr)'||'\cdot||pr'|'|'$. Let $q = r = p|'||$.

Case II. L affirmative., M, N negative.

Of the valid moods we prove only the critical cases:

$Z = |(pq)'|'\cdot|(qr)'|\cdot|pr'|' = |p'|'\cdot|q'|'\cdot|r'|\cdot|pr'|'$

1. $pr = o$, $Z = o$.

2. $pr' = o$, $Z = o$, for we shall assume:

$p \angle r\cdot \angle \cdot|p'|' \angle |r'|'$

This is not a general formula, i. e. one in which any value may be substituted for p and r. It is only true when $p \angle r$ is accepted categorically.

3. $p'r' = 0$, $Z = 0$

4. $|pr'|' = |p|' + |r'|' + pr \cdot |p'| + p'r' \cdot |r|$ and $Z = 0$.

$Z = |(pq)'|' \cdot |qr|' \cdot |(pr')| = |p'|' \cdot |q'|' \cdot |r| \cdot |qr|'$

1. $qr = 0$, $Z = 0$, for we assume as above

   $q \angle r' \cdot \angle \cdot |q'|' \angle |r|'$

2. $qr' = 0$, $Z = 0$.

3. $q'r = 0$, $Z = 0$.

4. $|qr|' = |q|' + |r|' + r'q \cdot |q'| + q'r \cdot |r'|$ and $Z = 0$.

$Z = ||(pq)'||' \cdot ||(qr)'|| \cdot ||pr'||' = ||p'||' \cdot ||q'||' \cdot ||r'|| \cdot ||pr'||'$

1. $pr = 0$, $Z = 0$.

2. $pr' = 0$, $Z = ||p'||' \cdot ||q'||' \cdot ||r'|| = 0$,

   for we shall assume in the same sense

   $p \angle r \cdot \angle \cdot ||p'||' \angle ||r'||'$

3. $p'r' = 0$, $Z = 0$.

4. $$\begin{aligned} ||pr'||' = {} & ||p|| + ||r'||' + p' \cdot |p| \cdot |r'|' + r \cdot |r'| \cdot |p|' \\ & + pr \cdot |r'| \cdot ||p'|'|' + r'p' \cdot |p| \cdot ||r|'|' \\ & + p \cdot |r'| \cdot |p'| \cdot ||r'|'| + r' \cdot |p| \cdot |r| \cdot ||p|'| \\ & + |p|' \cdot |r|' \cdot ||r|| + |r'|' \cdot |p'|' \cdot |p'|| \end{aligned}$$

   and $Z = 0$.

$Z = ||(pq)'||' \cdot ||qr||' \cdot ||(pr')'|| = ||p'||' \cdot ||q'||' \cdot ||r|| \cdot ||qr||'$

1. $qr = 0$, $Z = 0$, for we shall assume as before and in the same sense, $q \angle r' \cdot \angle \cdot ||q'||' \angle ||r||'$

2. $qr' = 0$, $Z = 0$.

3. $q'r = 0$, $Z = 0$.

4. $$\begin{aligned} ||qr||' = {} & ||q||' + ||r||' + q' \cdot |q| \cdot |r|' \cdot + r' \cdot |r| \cdot |q|' \\ & + qr' \cdot |r| \cdot ||q'|'|' + rq' \cdot |q| \cdot ||r'|'|' \\ & + q \cdot |r| \cdot |q'| \cdot ||r|'| + r \cdot |q| \cdot |r'| \cdot ||q|'| \\ & + |q|' \cdot |r'|' \cdot ||r'|| + |r|' \cdot |q'|' \cdot ||q'|| \end{aligned}$$

   and $Z = 0$.

$$pq \cdot (qr)' \cdot (pr')' = pq \cdot [q' + r'] \cdot [p' + r] = 0.$$

The invalidity of these members of the set that are untrue follows by one of the substitutions given below. We enumerate only the cases that are critical.

$\|\|pq\|'\|'\cdot\|\|qr\|\|'\cdot\|\|pr'\|\|'$	Let $q = \|\|p\|'\|'$, $r = \|p\|$
$pq\cdot\|\|(qr)'\|'\|'\cdot\|\|pr'\|\|'$	$p = r$
$pq\cdot\|\|qr\|\|'\cdot\|\|(pr')'\|'\|'$	$q = r'$
$\|\|(pq)'\|'\|\cdot\|\|(qr)'\|'\|'\cdot\|\|pr'\|\|'$	$p = q$, $pr' = 0$ and expanding the remaining factor.
$\|\|(pq)'\|'\|\cdot\|\|qr\|\|'\cdot\|\|(pr')'\|'\|'$	$p = q$, $pr = 0$ and expanding the remaining factor.
$\|(pq)'\|'\cdot\|\|(qr)'\|'\|'\cdot\|\|pr'\|'\|$	Expanding and multiplying out.
$\|(pq)'\|'\cdot\|\|qr\|'\|\cdot\|\|(pr')'\|'\|'$	Expanding and multiplying out.
$(pq)'\|'\cdot\|\|(qr)'\|\|\cdot\|\|pr'\|\|'$	$p = r$
$\|(pq)'\|'\cdot\|\|qr\|\|'\cdot\|\|(pr')'\|\|$	$q = r'$
$\|\|(pq)'\|\|'\cdot\|\|(qr)'\|\|\cdot\|pr'\|'$	$p = q = i$
$\|\|(pq)'\|\|'\cdot\|qr\|'\cdot\|\|(pr')'\|\|$	$p = q = i$

Case III. L, M and N affirmative.

The one hundred and fifty-two valid moods of this array follow at once from the examples that we prove below.

$Z = \|\|pq\|\|\cdot qr.\ pr' = 0.$

$Z = pq\cdot\|qr\|\cdot\|(pr')'\|' = q\cdot\|p'\|'\cdot\|r\|'\cdot\|qr\|$

1. $qr = 0$, $Z = 0$.
2. $qr' = 0$, $Z = q\cdot\|p'\|'\cdot\|r\|'\cdot = 0$, for
   $q \angle r\cdot r \angle \|r\|\cdot \angle \cdot q\cdot\|r\|' \angle 0$ $(q, r \neq 0, q, r \neq i)$
   If $q = 0$ or $r = 0$, $Z = 0$.
   If $q = i$, $r = 0$, $Z = 0$.
   If $r = i$, $Z = 0$.
3. $q'r = 0$, $Z = 0$.
4. $q'r' = 0$ or if none of the quadrants vanishes,
   $\|qr\| = qr + q'r'\cdot\|q\|\cdot\|r\| + \|q\|\cdot\|r'\|' + \|r\|\cdot\|q'\|'$
   and $Z = 0$.

$Z = pq\cdot\|(qr)'\|'\cdot\|pr'\| = p\cdot\|q'\|'\cdot\|r'\|'\cdot\|pr'\|$

1. $pr' = 0$, $Z = 0$.
2. $pr = 0$, $Z = p\cdot\|r'\|'\cdot\|q'\|' = 0$, for
   $p \angle r'\cdot r' \angle \|r'\|\cdot \angle \cdot p\cdot\|r'\|' \angle 0$ $(p, r \neq 0, p, r \neq i)$
   If $p = i$, $Z = 0$. If $p = 0$, $Z = 0$.
   If $r = i$, $p = 0$, $Z = 0$. If $r = 0$, $Z = 0$.

3. $p'r' = o$, $Z = o$.
4. $p'r = o$ or if none of the quadrants vanishes,
$|pr'| = pr' + p'r \cdot |p| \cdot |r'| + |p| \cdot |r|' + |r'| \cdot |p'|'$
and $Z = o$.

$Z = pq \cdot ||qr|| \cdot ||(pr')'||' = q \cdot ||qr|| \cdot ||p'||' \cdot ||r||'$

1. $qr = o$, $Z = o$.
2. $qr' = o$, $Z = q \cdot ||p'||' \cdot ||r||' = o$, for
$q \angle r \cdot r \angle ||r|| \cdot \angle \cdot q \cdot ||r||' \angle o$, $(q, r \neq o, q, r \neq i)$
3. $q'r = o$, $Z = q \cdot ||r|| \cdot ||p'||' \cdot ||r||' = o$.
4. $q'r' = o$ or if none of the quadrants of the universe vanishes,
$||qr|| = |qr| + |q|' \cdot |r|' \cdot ||q|| \cdot ||r||$
$+ q \cdot |q'| \cdot |r|' \cdot ||r|| + r \cdot |r'| \cdot |q|' \cdot ||q||$
$+ ||q'|'| \cdot ||r|'| + ||r'|'| \cdot ||q'|'$
$+ ||q|| \cdot ||r'||' + ||r|| \cdot ||q'||'$
and $Z = o$.

$Z = pq \cdot ||(qr)'||' \cdot ||pr'|| = p \cdot ||q'||' \cdot ||r'||' \cdot ||pr'||$

1. $pr' = o$, $Z = o$.
2. $pr = o$, $Z = p \cdot ||q'||' \cdot ||r'||' = o$, since
$pr \angle o \cdot \angle \cdot p \cdot ||r'||' \angle o$, $(p, r \neq o, p, r \neq i)$
3. $p'r' = o$, $Z = o$.
4. $p'r = o$ or if none of the quadrants vanishes,
$||pr'|| = pr'$ + etc. and $Z = o$ as before.

The invalid moods of the array all follow by means of a characteristic substitution. We enumerate only the instances which are the key to all the others.

$||pq'|' \cdot ||qr|'|' \cdot ||(pr')'||'$

Assume $p = i$ and $q \angle r$ and let $r = ||q|'|'$ where n is taken large enough to insure invalidity here and in cases of higher modal order. This choice is always possible because we have from our original induction formula that $q \angle ||q'|'_n$.

$||(pq)'||' \cdot ||qr|'|' \cdot |(pr')'|'|$

Assume $q = i$ and $p \angle r'$ and let $p = ||r|'|_n$ where n is 4 or larger.

$||(pq)'||' \cdot ||(qr)'|'| \cdot ||pr'|'|'$

Assume $p = i$ and $q = r$ and $q = ||r'|'|_n$.

$||pq|'|'\cdot||(qr)'||'\cdot||pr'|'|'$

Assume $q = i$ and $p \angle r'$ and $r' = ||p|'|'$.

In all of the following cases substitute $p = q = i$.

$||(pq)'||'\cdot||qr||\cdot|(pr')'|'$
$||(pq)'||'\cdot|qr|\cdot||(pr')'|'|$
$||(pq)'||'\cdot||qr|'|'\cdot pr'$
$||(pq)'||'\cdot qr\cdot||pr'|'|'$
$||(pq)'||'\cdot||(qr)'|'|\cdot|pr'|$
$||(pq)'||'\cdot|(qr)'|'\cdot||pr'||$

The array $Lpq.\ Mqr.\ Np'r' \angle o$

Affirmative	Negative	Invalid by the substitution
L, M, N		
	L, M, N	$p = q' = r = i$
L, M	N	$p = q = r = i$
L, N	M	
M, N	L	
L	M, N	$p = q = r' = i$
M	L, N	$p' = q = r = i$
N	L, M	$p' = q = r' = i$

Case I. L, M and N all affirmative.

The valid moods, one hundred and sixty in number, are true when the variables take on either of the limiting values zero or one. Excluding, then, these values, all follow from the two simple cases set down below.

$||(pq)'||'\cdot qr\cdot p'r' = o$
$pq\cdot||qr||\cdot p'r' = o$

The invalidity of all the others follows from one of two critical cases:

$||pq|'|'\cdot||qr|'|'\cdot||(p'r')'||'$
Suppose one possible case, $q \angle p$, $q \angle r$, $p \angle r$ and substitute $r = ||||q|'|||'\cdot$
$||(pq)'||'\cdot||(qr)'||'\cdot||p'r'|'|'$

Suppose one possible case, $q \angle p$, $q \angle r$, $p \angle r$ and substitute $q = ||||r'|'|||$.

Thus, for the first case,

$q \angle ||||q||'||'\cdot \therefore ||q|'| \angle ||||||q|'|||'||'\cdot \therefore ||||||q|'|||'|| \angle ||q|'|'\cdot$
$\therefore ||q|'|' \angle ||||||q|'|||'||$ is untrue.
$\therefore ||q|'|'\cdot||||||q|'|||'||' \angle o$ is untrue.

Case II. L and N affirmative, M negative.

The valid moods, one hundred and twelve in number, follow from

$pq\cdot||(qr)'||\cdot p'r' = o$

The invalid moods that remain follow from or in the same way as the two:

$||pq|'|'\cdot||qr||'\cdot||(p'r')'||'$

This becomes untrue for $r = o$, $q \angle p$, $p = |||q|'||'$

for $||||q||'|| \angle |q|'$
$\therefore ||||q|'||'|| \angle ||q|'|'$
$\therefore ||q|'|' \angle ||||q|'||'||$ is untrue
$\therefore ||q|'|'\cdot||||q|'||'||' \angle o$ is untrue
$||(pq)'||'\cdot||qr||'\cdot||p'r'|'|'$

This becomes untrue for $r = o$, $q \angle p$, $q = |||p'|'||$

for $p \angle |||p|'||'$
$\therefore ||p'|'| \angle |||||p'|'||'||'$
$\therefore |||||p'|'||'||' \angle ||p'|'|$ is untrue
$\therefore |||||p'|'||'||'\cdot||p'|'|' \angle o$ is untrue

Case III. M, N affirmative. L negative.

The valid moods, one hundred and fifty-four in number, are derived as follows, the instances given being the key to all the others.

$Z = ||(pq)'||\cdot qr\cdot p'r' = 0.$

$Z = ||(pq)'||\cdot|(qr)'|'\cdot|p'r'| =$

1. $pr' = 0$, $Z = 0$.
2. $p'r = 0$, $Z = ||(pq)'||\cdot|q'|'\cdot|r'|'\cdot|p'| = 0$,
   for $r \angle p\cdot \angle\cdot|r'|' \angle |p'|'$
3. $|p'r'| = p'r' + pr\cdot|p'|\cdot|r'| + |p'|\cdot|r|' + |r'|\cdot|p|'$
   and $Z = 0$.

We point out again that the implication accepted under 2 is only true when $r \angle p$ is accepted categorically. Otherwise we should have:

$r' \angle p'\cdot \angle\cdot|r|' \angle |p|'$, $(r = r', p = p')$ or
$p \angle r\cdot \angle\cdot|p| \angle |r|$, which is false for $r = ||p|'|'$

$Z = ||(pq)'||\cdot||(qr)'||'\cdot||p'r'|| = ||(pq)'||\cdot||q'||'\cdot||r'||'\cdot||p'r'||$

1. $pr' = 0$, $Z = 0$.
2. $p'r = 0$, $Z = 0$.
   $r \angle p\cdot \angle\cdot||r'||' \angle ||p'||'$
3. $||p'r'|| = |p'r'| + |p'|'\cdot|r'|'\cdot||p'||\cdot||r'||$
   $+ p'\cdot|p|\cdot|r'|'\cdot||r'|| + r'\cdot|r|\cdot|p'|'\cdot||p'||$
   $||p|'|\cdot||r'|'|' + ||r|'|\cdot||p'|'|'$
   $||p'||\cdot||r||' + ||r'||\cdot||p||'$
   and $Z = 0$.

$Z = ||(pq)'||\cdot||(qr)'|'|\cdot||p'r'|'|'$

1. $pr' = 0$, $Z = ||(pq)'||\cdot||(qr)'\imath'|\cdot||r'|'|'$
   1.1 $qr' = 0$, $Z = ||(pq)'||\cdot||q'|'|\cdot||r'|'|' = 0$

for we shall assume in the same sense as before

$q \angle r\cdot \angle\cdot||q'|'| \angle ||r'|'|$
1.2 $q'r = 0$, $Z = 0$.
1.3 $||(qr)'|'| = |q'|'\cdot|r'|' + |q'|\cdot|r'|\cdot||q'|'|\cdot||r'|'|$
$+ ||q'|'|\cdot||r'||' + ||r'|'|\cdot||q'||'$
and $Z = 0$.

2. $p'r = 0$, $Z = ||(pq)'||\cdot||(qr)'|'|\cdot||p'|'|'$
2.1 $qr' = 0$, $Z = ||(pq)'||\cdot||q'|'|\cdot||p'|'|' = 0$
$r = 0$ or $q = 0$, $Z = 0$. $p = 0$, $r = 0$.
$p = i$ or $r = i$, $Z = 0$. $q = i$, $r = i$.
$q \angle r\cdot r \angle p\cdot \angle \cdot q \angle p$ $(p, q, r \neq 0, p, q, r \neq i)$
$q \angle p\cdot \angle \cdot||q'|'| \angle ||p'|'|$
2.2 $q'r = 0$, $Z = ||(pq)'||\cdot||r'|'|\cdot||p'|'|' = 0$
for $r \angle p\cdot \angle \cdot||r'|'| \angle ||p'|'|$
2.3 $||(qr)'|'| = |q'|'\cdot|r'|' + |q'|\cdot|r'|\cdot||q'|'|\cdot||r'|'|$
$+ ||q'|'|\cdot||r'||' + ||r'|'|\cdot||q'||'$
and $Z = 0$, for $r \angle p\cdot \angle \cdot||r'|'| \angle ||p'|'|$ etc.
3. $||p'r'|'|' = |p|'\cdot|r|' + p'r'\cdot|p|\cdot|r| + pr\cdot||p'|'|'\cdot||r'|'|'$
$+ p\cdot|r|'\cdot||p'|'|' + r\cdot|p|'\cdot||r'|'|'$
$+ ||p'|'|'\cdot||r||' + ||r'|'|'\cdot||p||'$
3.1 $qr' = 0$, $Z = ||(pq)'||\cdot||q'|'|\cdot||p'r'|'|' = 0$
for $q \angle r\cdot \angle \cdot||q'|'|\cdot||r'|'|' \angle 0$
3.2 $q'r = 0$, $Z = ||(pq)'||\cdot||r'|'|\cdot||p'r'|'|' = 0$.
3.3 $||(qr)'|'| = |q'|'\cdot|r'|' +$ etc. and $Z = 0$.

The invalid moods of which

$||pq||'\cdot qr\cdot||p'r'|'|' \angle 0$
$||pq||'\cdot||(qr)'|'|\cdot|p'r'| \angle 0$
$||pq||'\cdot|(qr)'|'\cdot||p'r'|| \angle 0$

are the critical cases, are established by the substitution $p = q' = 0$, or as in the following example,

$||pq||'\cdot||(qr)'|'|\cdot||(p'r')'||' \angle 0$

Let $p = 0$ and suppose the case $q \angle r$ and let $r = |||q|'||'$

Thus, $||||q||'|| \angle |q|'$
$\therefore |||||q|'||'|| \angle ||q|'|'$
$\therefore ||q|'|' \angle |||||q|'||'|'|$ is untrue
$\therefore ||q|'|'\cdot|||||q|'||'|'|' \angle 0$ is untrue

The array $Lp'q.\ Mq'r.\ Nr'p \angle 0$

Affirmative	Negative	Invalid by the substitution
L, M, N		
	L, M, N	$p = q = r$
L, M	N	
L, N	M	Same case as the last
M, N	L	" " " " "
L	M, N	$p = q' = r' = o$
M	L, N	Same case as the last
N	L, M	" " " " "

Case I. L, M and N affirmative.

The valid moods, two hundred and eight in number, follow from one of the three simple cases given below:

$||p'q||\cdot q'r\cdot r'p \angle o$
$p'q\cdot||q'r||\cdot r'p \angle o$
$p'q\cdot q'r\cdot||r'p|| \angle o$

The invalidity of those moods that remain is established by a typical substitution. We cite the three critical instances:

$||p'q|'|'\cdot||q'r|'|'\cdot||(r'p)'||' \angle o$

Suppose the conditions $p \angle q'$, $q' \angle r$, $p \angle r$, and make

$p = |||q|'||$, $r = |||q'|'|'$.

$||p'q|'|'\cdot||(q'r)'||'\cdot||r'p|'|' \angle o$

Suppose the conditions $p' \angle q$, $r \angle p'$, $r \angle q$, and make

$q = |||p'|'|'$, $r = |||p|'||$.

$||(p'q)'||'\cdot||q'r|'|'\cdot||r'p|'|' \angle o$

Suppose the conditions $q \angle r$, $p \angle r'$, $p \angle q'$, and make

$q = |||p|'||$, $r = |||p|'||'\cdot||||p|'|||$.

Case II. L, M affirmative. N negative.

The valid moods, one hundred and twelve in number, follow from

$$p'q \cdot q'r \cdot ||(r'p)'|| = o.$$

The invalid moods follow from or in the same way as the two examples given below:

$$||p'q||'|' \cdot ||(q'r)'||' \cdot ||r'p||' \angle o$$

Suppose $p = o$, $q \angle r'$, and let $r = |||q|'||$.

$$||(p'q)'||' \cdot ||q'r|'|' \cdot ||r'p||' \angle o$$

Suppose $p = o$, $q \angle r'$ and let $q = |||r|'||$.

Our general method enables us to determine the truth or untruth of any proposition taken at random. The present paper takes the first steps in actually isolating the forms of those implications that are valid in a most general logic.

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